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Cyclic Homology of Filtered Algebras

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Abstract. We prove an analogue of a theorem of Quillen about the cyclic homology of filtered rings.

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1. Introduction

Quillen proves the following theorem about the algebraic K-theory of filtered rings:

THEOREM 1.1 ([4]), p. 112, Theorem 7). *Let A be a ring equipped with an increasing filtration F_pA, and such that F_0A is regular. Suppose that $B = gr(A)$ has finite Tor dimension as a right F_0A module and that F_0A has finite Tor dimension as a right B module. Then the inclusion $F_0A \subset A$ induces an isomorphism $K_i(F_0A) \cong K_i(A)$.*

This result implies for example that the natural inclusion $\mathcal{O}(V) \hookrightarrow \mathcal{D}(V)$ of the coordinate ring of a smooth affine variety into its ring of differential operators induces an isomorphism $K_i(\mathcal{O}(V)) \cong K_i(\mathcal{D}(V))$ in algebraic K-theory.

Recently, this calculation has been carried out in cyclic homology. Thus Block [1], Brylinski [2], and Wodzicki [5] show that one has an isomorphism in cyclic homology $HC_q(\mathcal{O}(V)) \cong HC_q(\mathcal{D}(V))$ for $q \geqslant 2n - 1$ where $\dim V = n$. This suggests that a theorem analogous to that of Quillen should hold. The purpose of this note is to prove such a theorem.

2. Notation

For convenience, A will always be an algebra over a field k of characteristic zero. By a filtration on an algebra we will mean an increasing filtration

$$0 = F_{-1}A \subset F_0A \subset F_1A \subset \dots$$

which satisfies $F_pA \cdot F_qA \subset F_{p+q}A$, $1 \in A$ and $A = \cup F_pA$. Hochschild homology of A with coefficients in a bimodule M will be denoted by $H_i(A; M)$. Cyclic homology of A is denoted by $HC_i(A)$ and periodic cyclic homology by $HP_i(A)$.

3. Cyclic Homology of Filtered Rings

DEFINITION 3.1. Let A be an algebra. Let $d(A) = \inf\{n \in \mathbf{Z}^+ \mid H_i(A; A) = 0$ for $i > n\}$. Call $d(A)$ the Hochschild dimension of A.

Remark 3.2. $d(A) = n$ implies that $S\colon HC_{i+2}(A) \to HC_i(A)$ is an isomorphism for all $i \geqslant n$, and an injection for $i = n - 1$. Thus $HP_i(A) \cong HC_i(A)$ for $i \geqslant n$.

LEMMA 3.3. $d(A) \leqslant d(gr(A))$

Proof. This is just a spectral sequence argument. □

We can now prove

THEOREM 3.4. *Let A be an algebra with an increasing filtration as above. Suppose that the Hochschild dimension of the associated graded algebra $d(grA) = n < \infty$. Then the natural map*

$$HC_i(F_0 A) \to HC_i(A)$$

is an isomorphism for all $i \geqslant n$. In particular $HP_i(F_0 A) \cong HP_i(A)$ for all i.

Proof. Let C_*A be the standard cyclic vector space associated to an algebra A, [3], p. 189. Thus $C_n A = A^{\otimes n+1}$ for each $n \geqslant 0$ and the face and degeneracy maps are defined as [*loc. cit.*]. Filter $C_n A$ in the following manner:

$$F_p C_n A = \Sigma_{k_0+k_1+\cdots+k_n=p} F_{k_0} A \otimes \cdots \otimes F_{k_n} A.$$

It is easy to see that the face, degeneracy and cyclic maps preserve this filtration, hence, for each p, $F_p C_* A$ forms a cyclic vector space.

LEMMA 3.5. $gr(F_* C_* A) \cong C_* gr(A)$.

LEMMA 3.6

$$S: HC_{i+2}(F_l C_* A/F_{l-1} C_* A) \to HC_i(F_l C_* A/F_{l-1} C_* A)$$

acts like zero.

Proof. As in [3], p. 197, claim 1, define on gr(A) a derivation δ: gr(A) → gr(A) by the following formula $\delta(a) = |a|a$ for homogeneous elements of degree $|a|$. Then define the endomorphism $\mathscr{L}_\delta$ of C_*A by

$$\mathscr{L}_\delta(a_0 \otimes \cdots \otimes a_n) = \Sigma_{i=0}^n a_0 \otimes \cdots \otimes \delta(a_i) \otimes \cdots \otimes a_n.$$

Then by [3] $\mathscr{L}_\delta \circ S = 0$ on HC_*(gr(A)). On the other hand it is clear that $\mathscr{L}_\delta$ acts by multiplication by l on $HC_*(F_l C_* A/F_{l-1} C_* A)$. So that $S: HC_{i+2}(F_l C_* A/F_{l-1} C_* A) \to HC_i(F_l C_* A/F_{l-1} C_* A)$ acts like zero, for $l > 0$. □

To continue with the proof of the theorem consider the following short exact sequence of cyclic vector spaces, $0 \to F_0 C_* A \to F_1 C_* A \to F_1 C_* A/F_0 C_* A \to 0$. This induces a long exact sequence in cyclic homology

$$\cdots \to HC_i(F_0 C_* A) \to HC_i(F_1 C_* A) \to HC_i(F_1 C_* A/F_0 C_* A) \to \cdots$$

By Lemma 3.6 above, S acts like zero on $HC_i(F_1 C_* A/F_0 C_* A)$ while by the hypothesis on the Hochschild dimension,

$$S: HC_{i+2}(F_1 C_* A/F_0 C_* A) \to HC_i(F_1 C_* A/F_0 C_* A)$$

is an isomorphism for $i \geqslant n$, hence $HC_i(F_1 C_* A/F_0 C_* A) = 0$. Therefore by the long exact sequence above we know that $HC_i(F_0 C_* A) \to HC_i(F_1 C_* A)$ is an isomorphism, for $i \geqslant n$. Continuing in the same manner, it follows from the short exact sequence of cyclic vector spaces

$$0 \to F_l C_* A \to F_{l+1} C_* A \to F_{l+1} C_* A / F_l C_* A \to 0$$

that $HC_i(F_l C_* A) \to HC_i(F_{l+1} C_* A)$ is an isomorphism for $i \geqslant n$. We then have the chain of isomorphisms for $i \geqslant n$,

$$HC_i(F_0 A) = HC_i(F_0 C_* A) \xrightarrow{\sim} HC_i(F_1 C_* A) \xrightarrow{\sim} \cdots \xrightarrow{\sim} HC_i(F_l C_* A) \xrightarrow{\sim} \cdots$$

and the theorem follows by observing that $HC_i(A) = \lim HC_i(F_l C_* A)$ □

Examples: (1) Let $A = U(X)$ be the universal enveloping algebra of a Lie algebra X of dimension n over a field k of characteristic 0. Then $\mathrm{gr}(A) \cong S^*(X)$ and $d(S^*(X)) = n$. For $i \geqslant n$ $HC_i(A) \cong HC_i(F_0 A) = HC_i(k)$.

(2) Let $\mathscr{D}(V)$ be the algebra of differential operators on a smooth affine variety V. Then $\mathrm{gr}(\mathscr{D}(V)) \cong \mathcal{O}(T^*V)$ and $d(\mathcal{O}(T^*V)) = 2n$ so that $HC_i(\mathcal{O}(V)) \xrightarrow{\sim} HC_i(\mathscr{D}(V))$ for $i \geqslant 2n$.

(3) Let $Y \to X$ be the Karoubi–Jouanolou resolution of a smooth projective variety X, as in [2]. Thus, Y is a smooth affine variety and the fiber is affine space. Then Brylinski [2] shows that

$$H_*(\mathscr{D}_{Y/X}; \mathscr{D}_{Y/X}) \cong \oplus H^{2n+i-*}(X, \Omega^i_X).$$

is the Hodge cohomology. By the result above,

$$HP_*(\mathscr{D}_{Y/X}) \cong \oplus H^{2i+*}(Y) \cong \oplus H^{2i+*}(X)$$

is just the de Rham cohomology of X.

Remark 3.7 (1) It seems likely that one should be able to replace the hypothesis that $d(A) < \infty$ which is a statement about the homological dimension of A over k, by one about the homological dimension of A over $F_0 A$ as in Quillen's theorem.

(2) The theorem is true in the situation where one considers a topological algebra as long as one makes appropriate compatibility conditions on the filtration and the topology.

In [1], [2] and [5], the Hochschild homology of $\mathscr{D}(V)$ is computed and it is shown that

$$H_i(\mathscr{D}(V); \mathscr{D}(V)) \cong H^{2n-i}(V)$$

where this latter group is de Rham cohomology of V. Now our theorem immediately implies

COROLLARY 3.8. *The Hochschild to cyclic spectral sequence degenerates at the* E^1 *term. Thus*

$$HC_i(\mathscr{D}(V)) \cong H^{2n-i}(V) \oplus H^{2n-i+2}(V) \oplus \cdots$$

Proof. By the theorem above and the computation of the Hochschild homology, it follows that for $i \geqslant 2n$ that

$$\Sigma_{j+l=i} E^1_{jl} \cong \Sigma_{j+l=i} E^{\infty}_{jl}$$

hence any differential must be zero when the target of the differential has total degree greater than $2n$. But each differential, as for example, the one at the E^1 term which is just Connes' B operator, is given by an explicit formula which does not depend on where in the spectral sequence it lies. Hence, all the differentials must be zero. □

Remark 3.10. This shows that the theorem is best possible in one sense; in general, one can not have an isomorphism until one reaches the stable range.

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